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CERTAIN GEOMETRIC PROPERTIES OF ℓ-HYPERGEOMETRIC FUNCTION

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Abstract. Geometric function theory is the branch of complex analysis which deals with the geometric properties of analytic functions. It was founded around the turn of the 20th century and has remained one of the active fields of the current research. In this paper, we study certain geometric properties like κ-uniformly convexity and κ-starlikeness of ℓ-Hypergeometric function and then we prove Alexander transform of ℓ-Hypergeometric function is starlike.

Keywords: Univalent function, close-to-convex function, κ-uniformly iconvexity, κ-starlikeness, Alexander transform, ℓ-Hypergeometric ifunction.

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1 Introduction

Let $\mathcal{A}(\mathbb{D}_1(0))$ denote the class of analytic functions in the open unit disk $\mathbb{D}_1(0) = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{C}$ be the class of all functions $f \in \mathcal{A}(\mathbb{D}_1(0))$ which are normalized by $f(0) = 0$ and $f'(0) = 1$ and have the form (Maharana et al., 2018; Mehrez, 2019; Oluwayemi & Faisal, 2021; Ponnusamy et al., 2011; Ponnusamy & Vuorinen, 2001; Prajapat, 2014, 2011; Purohit, 2012; Vidyasagar, 2020)

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}_1(0). \tag{1}$$

Two functions $f, g \in \mathcal{A}(\mathbb{D}_1(0))$ we say that f is subordinated to g in $\mathbb{D}_1(0)$ and express symbolically $f(z) \prec g(z)$, if there exists a function $\omega \in \mathcal{A}(\mathbb{D}_1(0))$ with $|\omega(z)| < |z|$, $z \in \mathbb{D}_1(0)$ such that $f(z) = g(\omega(z))$ in $\mathbb{D}_1(0)$. Furthermore, if function f is univalent in $\mathbb{D}_1(0)$, then g is subordinate to f provided $g(0) = f(0)$ and $g(\mathbb{D}_1(0)) \subset f(\mathbb{D}_1(0))$. By $\mathcal{S}$ we denote the class of all functions in $\mathcal{C}$ which are univalent in $\mathbb{D}_1(0)$. Let $\mathcal{S}^*(\varepsilon)$, $\mathcal{C}(\varepsilon)$, $\mathcal{K}(\varepsilon)$, $\widetilde{\mathcal{S}}^*(\varepsilon)$ and $\widetilde{\mathcal{C}}(\varepsilon)$ denote the classes of starlike, convex, close-to-convex, strongly starlike and strongly convex functions of order ε, respectively, and are defined as

$$\mathcal{S}^*(\varepsilon) = \left\{ f \in \mathcal{C} : \mathcal{R}e\left(\frac{zf'(z)}{f(z)}\right) > \varepsilon,\ z \in \mathbb{D}_1(0),\ 0 \leq \varepsilon < 1 \right\},$$

$$\mathcal{C}(\varepsilon) = \left\{ f \in \mathcal{C} : \mathcal{R}e\left(\frac{\left(zf'(z)\right)'}{f'(z)}\right) > \varepsilon,\ z \in \mathbb{D}_1(0),\ 0 \leq \varepsilon < 1 \right\},$$

$$\mathcal{K}(\varepsilon) = \left\{ f \in \mathcal{C} : \mathcal{R}e\left(\frac{zf'(z)}{g(z)}\right) > \varepsilon,\ z \in \mathbb{D}_1(0),\ 0 \le \varepsilon < 1,\ g \in \mathcal{S}^*(0) \equiv \mathcal{S}^* \right\},$$

$$\widetilde{\mathcal{S}}^*(\varepsilon) = \left\{ f \in \mathcal{C} : \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\varepsilon\pi}{2},\ z \in \mathbb{D}_1(0),\ 0 \le \varepsilon < 1 \right\},$$

and

$$\widetilde{\mathcal{C}}(\varepsilon) = \left\{ f \in \mathcal{C} : \left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\varepsilon\pi}{2},\quad z \in \mathbb{D}_1(0),\quad 0 \le \varepsilon < 1 \right\}.$$

For more details regarding these classes see Duren (1983); Goodman (1983).

For $z \in \mathbb{C}$, the ℓ-Hypergeometric function is defined as

$$H\begin{bmatrix} a; & & z \\ & b;\ (c:\ell); & \end{bmatrix} = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n (c)_n^{\ell n}} \frac{z^n}{n!}, \tag{2}$$

where $(\gamma)_n = \Gamma(\gamma+n)/\Gamma(\gamma)$, $a,\ \ell \in \mathbb{C}$ with $\mathcal{R}e(\ell) \ge 0$ and $b,\ c \in \mathbb{C}\backslash\{0,-1,-2,\cdots\}$. If we put $\ell = 0$ in (2), then ℓ-H function turns to well known confluent hypergeometric function,

$$H\begin{bmatrix} a; & & z \\ & b;\ (c:0); & \end{bmatrix} = {}_1F_1\begin{bmatrix} a; & z \\ b; & \end{bmatrix}. \tag{3}$$

The ℓ-H function (2) was recently studied in Chudasama & Dev (2016).

We note that ℓ-Hypergeometric function (2) does not belong to the family $\mathcal{C}$. Thus, it is natural to consider the following normalization of ℓ-H function:

$$\begin{aligned} \mathcal{H}(a;b;(c,\ell);z) &= zH\begin{bmatrix} a; & & z \\ & b;\ (c:\ell); & \end{bmatrix} \\ &= z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(b)_{n-1}(c)_{n-1}^{\ell(n-1)}} \frac{z^n}{(n-1)!}. \end{aligned} \tag{4}$$

Motivated by above works, in this paper we study certain geometric properties like κ-uniformly convexity and κ-starlikeness of ℓ-Hypergeometric function and then we prove Alexander transform of ℓ-Hypergeometric function is starlike. Let $\kappa - \mathcal{UCV}$ and $\kappa - \mathcal{ST}$ be the subclasses of $\mathcal{S}$ consisting of functions which are κ-uniformly convex and κ-starlike, respectively (Kanas & Wisniowska, 1999, 2000). They are given by,

$$\kappa - \mathcal{UCV} = \left\{ f \in \mathcal{S} : \mathcal{R}e\left(1 + \frac{zf''(z)}{f'(z)}\right) > \kappa\left|\frac{zf''(z)}{f'(z)}\right|,\ z \in \mathbb{D}_1(0), \kappa \ge 0 \right\}, \tag{5}$$

$$\kappa - \mathcal{ST} = \left\{ f \in \mathcal{S} : \mathcal{R}e\left(\frac{zf'(z)}{f(z)}\right) > \kappa\left|\frac{zf'(z)}{f(z)} - 1\right|,\ z \in \mathbb{D}_1(0), \kappa \ge 0 \right\}. \tag{6}$$

The class of all functions $p \in \mathcal{A}(\mathbb{D}_1(0))$ with $p(0) = 1$ satisfying the condition

$$\mathcal{R}e\, p(z) > \varepsilon,\ z \in \mathbb{D}_1(0),\ \varepsilon \in [0,1)$$

be denoted by $\mathcal{P}(\varepsilon)$. In particular, $\mathcal{P}(0) = \mathcal{P}$ is the well-known Carathe$\acute{o}$dory class of functions with positive real part in $\mathbb{D}_1(0)$ (Goodman, 1983). The following lemmas are useful in the next section.

Lemma 1. *(Owa et al., 2002) If $f \in \mathcal{C}$ satisfies the inequality*

$$|zf''(z)| < \frac{1-\varepsilon}{4}, \ z \in \mathbb{D}_1(0), \ \varepsilon \in [0,1), \tag{7}$$

then,

$$\mathcal{R}ef'(z) > \frac{1+\varepsilon}{2}, \ z \in \mathbb{D}_1(0), \ \varepsilon \in [0,1).$$

Lemma 2. *(Silverman, 1975) Let $f \in \mathcal{C}$ and $\varepsilon \in [0,1)$, then*

(i) $f \in \mathcal{S}^*(\varepsilon)$ provided

$$\sum_{n=2}^{\infty}(n-\varepsilon)|a_n| \leq 1-\varepsilon \tag{8}$$

(i) $f \in \mathcal{C}(\varepsilon)$ provided

$$\sum_{n=2}^{\infty}n(n-\varepsilon)|a_n| \leq 1-\varepsilon. \tag{9}$$

Lemma 3. *(Kanas & Wisniowska, 1999, 2000) Let $f \in \mathcal{C}$. If for some $\kappa \geq 0$,*

$$\sum_{n=2}^{\infty}n(n-1)|a_n| \leq \frac{1}{\kappa+2} \tag{10}$$

and

$$\sum_{n=2}^{\infty}[n+\kappa(n-1)]|a_n| \leq 1, \tag{11}$$

then $f \in \kappa-\mathcal{UCV}$ and $f \in \kappa-\mathcal{ST}$, respectively.

2 Main Results

In the sequence, convexity of order ε, close-to-convexity of order $(1+\varepsilon)/2$ for normalized ℓ-Hypergeometric function $\mathcal{H}(a;b;(c,\ell);z)$ are investigated. Certain sufficient conditions for $\mathcal{H}(a;b;(c,\ell);z)$ to be in the classes $\mathcal{P}(\varepsilon), \mathcal{S}^*(\varepsilon)$, $\mathcal{C}(\varepsilon), \kappa-\mathcal{UCV}$ and $\kappa-\mathcal{ST}$ are also given.

Theorem 1. *If $a,b,c,\ell \in \mathbb{R}$ with $0 < a < b$, $\ell \geq 1$ and $c \geq 1+\sqrt{3}$. Then $\mathcal{H}(a;b;(c,\ell);z)$ is starlike in $\mathbb{D}_1(0)$ i.e $\mathcal{H}(a;b;(c,\ell);\cdot) \in \mathcal{S}^*$.*

Proof. Let $p(z)$ be the function defined by

$$p(z) = \frac{z\mathcal{H}'(a;b;(c,\ell);z)}{\mathcal{H}(a;b;(c,\ell);z)}, \quad z \in \mathbb{D}_1(0).$$

Since $\dfrac{\mathcal{H}(a;b;(c,\ell);z)}{z} \neq 0$, the function p is analytic in $\mathbb{D}_1(0)$ and $p(0)=1$. To prove the result, we need to show that $\mathcal{R}e(p(z)) > 0$. Since $c > 1$ and $\ell \geq 1$, it follows that $(c)_n \leq (c)_n^{\ell n}$ for all $n \in \mathbb{N}$. So, from the hypothesis,

$$\begin{aligned}\left|\mathcal{H}'(a;b;(c,\ell);z) - \frac{\mathcal{H}(a;b;(c,\ell);z)}{z}\right| &= \left|\sum_{n=1}^{\infty}\frac{(a)_n}{(b)_n(c)_n^{\ell n}}\frac{z^n}{(n-1)!}\right| \\ &< \sum_{n=1}^{\infty}\frac{1}{(c)_n} \\ &= \sum_{n=1}^{\infty}\frac{1}{c(c+1)(c+2)\cdots(c+n-1)} \\ &< \frac{1}{c}\sum_{n=0}^{\infty}\frac{1}{(c+1)^n} = \frac{c+1}{c^2},\end{aligned} \tag{12}$$

and

$$\begin{aligned}\left|\frac{\mathcal{H}(a;b;(c,\ell);z)}{z}\right| &\geq 1-\left|\sum_{n=1}^{\infty}\frac{(a)_n}{(b)_n(c)_n^{\ell n}}\frac{z^n}{n!}\right| \\ &\geq 1-\sum_{n=1}^{\infty}\frac{1}{(c)_n} \\ &= 1-\sum_{n=1}^{\infty}\frac{1}{c(c+1)(c+2)\cdots(c+n-1)} \\ &> 1-\frac{1}{c}\sum_{n=0}^{\infty}\frac{1}{(c+1)^n}=\frac{c^2-c-1}{c^2}.\end{aligned} \tag{13}$$

From (12) and (13), we have

$$\begin{aligned}\left|\frac{z\mathcal{H}'(a;b;(c,\ell);z)}{\mathcal{H}(a;b;(c,\ell);z)}-1\right| &= \left|\frac{\mathcal{H}'(a;b;(c,\ell);z)-\frac{\mathcal{H}(a;b;(c,\ell);z)}{z}}{\frac{\mathcal{H}(a;b;(c,\ell);z)}{z}}\right| \\ &< \frac{c+1}{c^2-c-1}, \quad z\in\mathbb{D}_1(0).\end{aligned}$$

Since $c\geq 1+\sqrt{3}$, it follows that $\frac{c+1}{c^2-c-1}\leq 1$ and hence $\mathcal{H}(a;b;(c,\ell);z)$ is starlike in $\mathbb{D}_1(0)$. □

Theorem 2. *If $a,b,c,\ell\in\mathbb{R}$ with $0<a<b$, $\ell\geq 1$. For $0\leq\varepsilon<1$, let*

$$\varrho(\varepsilon)=\frac{(2-\varepsilon)+\sqrt{5\varepsilon^2-16\varepsilon+12}}{2(1-\varepsilon)}.$$

If $c\geq\varrho(\varepsilon)$, then $\mathcal{H}(a;b;(c,\ell);z)$ is starlike function of order ε i.e $\mathcal{H}(a;b;(c,\ell);\cdot)\in\mathcal{S}^*(\varepsilon)$.

Proof. Following the proof of Theorem 1, $\mathcal{H}(a;b;(c,\ell);z)$ is starlike function of order ε, if $\dfrac{c+1}{c^2-c-1}\leq 1-\varepsilon$. This is true from the hypothesis. This completes the proof. □

Theorem 3. *If $a,b,c,\ell\in\mathbb{R}$ with $0<a<b$, $\ell\geq 1$. For $0\leq\varepsilon<1$, let*

$$\vartheta(\varepsilon)=\frac{(8-3\varepsilon)+\sqrt{17\varepsilon^2-68\varepsilon+76}}{2(1-\varepsilon)}.$$

If $c\geq\vartheta(\varepsilon)$. Then $\mathcal{H}(a;b;(c,\ell);z)$ is convex of order ε i.e $\mathcal{H}(a;b;(c,\ell);\cdot)\in\mathcal{C}(\varepsilon)$.

Proof. Under the hypothesis, we obtain

$$\begin{aligned}\left|\mathcal{H}'(a;b;(c,\ell);z)\right| &\leq \left|1+\sum_{n=1}^{\infty}\frac{(a)_n}{(b)_n(c)_n^{\ell n}}\frac{(n+1)z^n}{n!}\right| \\ &\leq 1+\sum_{n=1}^{\infty}\frac{n+1}{(c)_n} \\ &= 1+\sum_{n=1}^{\infty}\frac{n}{(c)_n}+\sum_{n=1}^{\infty}\frac{1}{(c)_n} \\ &\leq 1+\frac{1}{c}+\frac{2}{c}\sum_{n=0}^{\infty}\frac{1}{(c+1)^n} \\ &= \frac{c^2+3c+2}{c^2}.\end{aligned} \tag{14}$$

For the reverse inequality, we have

$$\begin{aligned}
\left|\mathcal{H}'(a;b;(c,\ell);z)\right| &\geq 1-\left|\sum_{n=1}^{\infty}\frac{(a)_n}{(b)_n(c)_n^{\ell n}}\frac{(n+1)z^n}{n!}\right| \\
&\geq 1-\sum_{n=1}^{\infty}\frac{n+1}{(c)_n} \\
&= 1-\sum_{n=1}^{\infty}\frac{n}{(c)_n}-\sum_{n=1}^{\infty}\frac{1}{(c)_n} \qquad (15)\\
&\geq 1-\frac{1}{c}-\frac{2}{c}\sum_{n=0}^{\infty}\frac{1}{(c+1)^n} \\
&= \frac{c^2-3c-2}{c^2}.
\end{aligned}$$

From (14) and (15), we obtained

$$\frac{c^2-3c-2}{c^2} \leq \left|\mathcal{H}'(a;b;(c,\ell);z)\right| \leq \frac{c^2+3c+2}{c^2}, \; z \in \mathbb{D}_1(0). \qquad (16)$$

From (4), we have

$$\begin{aligned}
|z\mathcal{H}''(a;b;(c,\ell);z)| &= \left|\sum_{n=1}^{\infty}\frac{(a)_n}{(b)_n(c)_n^{\ell n}}\frac{n(n+1)z^n}{n!}\right| \\
&\leq \sum_{n=1}^{\infty}\frac{n(n+1)}{(c)_n} \qquad (17)\\
&\leq \frac{4}{c}+\frac{1}{c}\sum_{n=0}^{\infty}\frac{1}{(c+1)^n} \\
&= \frac{5c+1}{c^2}.
\end{aligned}$$

Now, from (16) and (17), we get

$$\left|\frac{z\mathcal{H}''(a;b;(c,\ell);z)}{\mathcal{H}'(a;b;(c,\ell);z)}\right| \leq \frac{5c+1}{c^2-3c-2}, \quad z \in \mathbb{D}_1(0).$$

Since $c > \vartheta(\varepsilon)$, it follows that $\dfrac{5c+1}{c^2-3c-2} \leq 1-\varepsilon$. Hence, $\mathcal{H}(a;b;(c,\ell);z)$ is convex of order ε in $\mathbb{D}_1(0)$. $\square$

If we take $\varepsilon = 0$ in Theorem 3, then we have the following result.

Corollary 1. *If $a,b,c,\ell \in \mathbb{R}$ with $0 < a < b$, $\ell \geq 1$ and $c \geq 4+\sqrt{19}$. Then $\mathcal{H}(a;b;(c,\ell);\cdot) \in \mathcal{C}$.*

Theorem 4. *If $a,b,c,\ell \in \mathbb{R}$ with $0 < a < b$, $\ell \geq 1$. For $0 \leq \varepsilon < 1$, let*

$$\rho(\varepsilon) = \frac{10+2\sqrt{26-\varepsilon}}{1-\varepsilon}.$$

If $c \geq \rho(\varepsilon)$. Then $\mathcal{H}(a;b;(c,\ell);z)$ is close-to-convex of order $\dfrac{1+\varepsilon}{2}$ i.e $\mathcal{H}(a;b;(c,\ell);\cdot) \in \mathcal{K}\left(\dfrac{1+\varepsilon}{2}\right)$.

Proof. Using (17) and Lemma 1, we have

$$|z\mathcal{H}''(a;b;(c,\ell);z)| \leq \frac{5c+1}{c^2}, \quad z \in \mathbb{D}_1(0).$$

Since $c \geq \rho(\varepsilon)$, it follows that $\dfrac{5c+1}{c^2} \leq \dfrac{1-\varepsilon}{4}$. this proves that $\mathcal{R}e(\mathcal{H}'(a;b;(c,\ell);z)) > \dfrac{1+\varepsilon}{2}$ and hence $\mathcal{H}(a;b;(c,\ell);\cdot) \in \mathcal{K}\left(\dfrac{1+\varepsilon}{2}\right)$. □

If we take $\varepsilon = 0$ in Theorem 4, then we have the following result.

Corollary 2. *If $a,b,c,\ell \in \mathbb{R}$ with $0 < a < b$, $\ell \geq 1$ and $c \geq 10 + 2\sqrt{26}$. Then $\mathcal{H}(a;b;(c,\ell);z)$ is close-to-convex of order $\frac{1}{2}$ i.e $\mathcal{H}(a;b;(c,\ell);\cdot) \in \mathcal{K}\left(\frac{1}{2}\right)$.*

Theorem 5. *If $a,b,c,\ell \in \mathbb{R}$ with $0 < a < b$, $\ell \geq 1$. For $0 \leq \varepsilon < 1$, let*

$$\psi(\varepsilon) = \frac{1+\sqrt{5-4\varepsilon}}{2(1-\varepsilon)}.$$

If $c \geq \psi(\varepsilon)$. Then $\dfrac{\mathcal{H}(a;b;(c,\ell);z)}{z} \in \mathcal{P}(\varepsilon)$.

Proof. Let $p(z)$ be the function defined by

$$p(z) = \frac{\mathcal{H}(a;b;(c,\ell);z)/z - \varepsilon}{(1-\varepsilon)}.$$

The function $p(z)$ is analytic in $\mathbb{D}_1(0)$ and $p(0) = 1$. To prove the result, we have to show that $|p(z) - 1| < 1$. If $z \in \mathbb{D}_1(0)$, then

$$\begin{aligned}
|p(z)-1| &= \left| \frac{1}{1-\varepsilon} \sum_{n=1}^{\infty} \frac{(a)_n}{(b)_n (c)_n^{\ell n}} \frac{z^n}{n!} \right| \\
&\leq \frac{1}{1-\varepsilon} \sum_{n=1}^{\infty} \frac{1}{(c)_n} \\
&= \frac{1}{1-\varepsilon} \sum_{n=1}^{\infty} \frac{1}{c(c+1)(c+2)\cdots(c+n-1)} \\
&\leq \frac{1}{1-\varepsilon} \frac{1}{c} \sum_{n=0}^{\infty} \frac{1}{(c+1)^n} = \frac{c+1}{c^2(1-\varepsilon)}.
\end{aligned}$$

Since $c \geq \psi(\varepsilon)$, it follows that $\dfrac{c+1}{c^2(1-\varepsilon)} \leq 1$. Hence, $\dfrac{\mathcal{H}(a;b;(c,\ell);z)}{z} \in \mathcal{P}(\varepsilon)$. □

If we take $\varepsilon = 0$ in Theorem 5, then we have the following result.

Corollary 3. *If $a,b,c,\ell \in \mathbb{R}$ with $0 < a < b$, $\ell \geq 1$ and $c \geq \dfrac{1+\sqrt{5}}{2}$. Then $\dfrac{\mathcal{H}(a;b;(c,\ell);z)}{z} \in \mathcal{P}$.*

Theorem 6. *If $a,b,c,\ell \in \mathbb{R}$ with $0 < a < b$, $c \geq 1$ and $\ell \geq 1$. For $0 \leq \varepsilon < 1$,*

$$\mathcal{H}'(a;b;(c,\ell);1) - \varepsilon\mathcal{H}(a;b;(c,\ell);1) \leq 2(1-\varepsilon).$$

Then $\mathcal{H}(a;b;(c,\ell);z) \in \mathcal{S}^*(\varepsilon)$.

Proof. From (4), we have $\mathcal{H}(a;b;(c,\ell);z) = z + \sum_{n=2}^{\infty} A_{n-1} z^n$, where $A_{n-1} = \dfrac{(a)_{n-1}}{(n-1)!(b)_{n-1}(c)_{n-1}^{\ell(n-1)}}$. Then from the hypothesis, we have

$$\begin{aligned}
\sum_{n=2}^{\infty} (n-\varepsilon)\,|A_{n-1}| &= \sum_{n=2}^{\infty} nA_{n-1} - \varepsilon \sum_{n=2}^{\infty} A_{n-1} \\
&= (\mathcal{H}'(a;b;(c,\ell);z) - 1) - \varepsilon(\mathcal{H}(a;b;(c,\ell);z) - 1) \\
&= \mathcal{H}'(a;b;(c,\ell);z) - \varepsilon\mathcal{H}(a;b;(c,\ell);z) - 1 + \varepsilon \\
&\leq 1 - \varepsilon.
\end{aligned}$$

Hence form Lemma 2, $\mathcal{H}(a;b;(c,\ell);z)$ is a starlike of order ε. $\square$

Theorem 7. *If $a,b,c,\ell \in \mathbb{R}$ with $0<a<b$, $c \geq 1$ and $\ell \geq 1$. For $0 \leq \varepsilon < 1$,*

$$\mathcal{H}''(a;b;(c,\ell);1) + (1-\varepsilon)\mathcal{H}'(a;b;(c,\ell);1) \leq 2(1-\varepsilon).$$

Then $\mathcal{H}(a;b;(c,\ell);z) \in \mathcal{C}(\varepsilon)$.

Proof. From (4), we have $\mathcal{H}(a;b;(c,\ell);z) = z + \sum_{n=2}^{\infty} A_{n-1} z^n$, where $A_{n-1} = \dfrac{(a)_{n-1}}{(n-1)!(b)_{n-1}(c)_{n-1}^{\ell(n-1)}}$. Then from the hypothesis, we have

$$\begin{aligned}
\sum_{n=2}^{\infty} n(n-\varepsilon)\,|A_{n-1}| &= \sum_{n=2}^{\infty} n(n-1)A_{n-1} + (1-\varepsilon)\sum_{n=2}^{\infty} A_{n-1} \\
&= \mathcal{H}''(a;b;(c,\ell);1) + (1-\varepsilon)(\mathcal{H}'(a;b;(c,\ell);1) - 1) \\
&= \mathcal{H}''(a;b;(c,\ell);1) + (1-\varepsilon)\mathcal{H}'(a;b;(c,\ell);1) - (1-\varepsilon) \\
&\leq 1-\varepsilon.
\end{aligned}$$

Hence form Lemma 2, $\mathcal{H}(a;b;(c,\ell);z)$ is a convex of order ε. $\square$

Theorem 8. *If $a,b,c,\ell \in \mathbb{R}$ with $0<a<b$, $c \geq 1$, $\ell \geq 1$ and $\kappa \geq 0$. Then the sufficient condition for $\mathcal{H}(a;b;(c,\ell);z)$ to be in $\kappa - \mathcal{ST}$ is*

$$\mathcal{H}'(a;b;(c,\ell);1) - \frac{\kappa}{\kappa+1}\mathcal{H}(a;b;(c,\ell);1) \leq \frac{2}{\kappa+1}.$$

Proof. From (4), we have $\mathcal{H}(a;b;(c,\ell);z) = z + \sum_{n=2}^{\infty} A_{n-1} z^n$, where $A_{n-1} = \dfrac{(a)_{n-1}}{(n-1)!(b)_{n-1}(c)_{n-1}^{\ell(n-1)}}$. Then from the hypothesis, we have

$$\begin{aligned}
\sum_{n=2}^{\infty} [n+\kappa(n-1)]\,|A_{n-1}| &= (1+\kappa)\sum_{n=2}^{\infty} nA_{n-1} - \kappa\sum_{n=2}^{\infty} A_{n-1} \\
&= (1+\kappa)(\mathcal{H}'(a;b;(c,\ell);z) - 1) \\
&\qquad - \kappa(\mathcal{H}(a;b;(c,\ell);z) - 1) \\
&= (1+\kappa)\mathcal{H}'(a;b;(c,\ell);z) \\
&\qquad - \kappa\mathcal{H}(a;b;(c,\ell);z) - 1 \\
&\leq 1.
\end{aligned}$$

Hence form Lemma 3, $\mathcal{H}(a;b;(c,\ell);z) \in \kappa - \mathcal{ST}$. $\square$

Theorem 9. *If $a,b,c,\ell \in \mathbb{R}$ with $0<a<b$, $c \geq 1$, $\ell \geq 1$ and $\kappa \geq 0$. Then the sufficient condition for $\mathcal{H}(a;b;(c,\ell);z)$ to be in $\kappa - \mathcal{UCV}$ is*

$$\mathcal{H}''(a;b;(c,\ell);1) \leq \frac{1}{\kappa+2}.$$

Proof. From (4), we have $\mathcal{H}(a;b;(c,\ell);z) = z + \sum_{n=2}^{\infty} A_{n-1} z^n$, where $A_{n-1} = \dfrac{(a)_{n-1}}{(n-1)!(b)_{n-1}(c)_{n-1}^{\ell(n-1)}}$. Then from the hypothesis, we have

$$\begin{aligned}
\sum_{n=2}^{\infty} n(n-1)\,|A_{n-1}| &= \sum_{n=2}^{\infty} n(n-1)A_{n-1} \\
&= \mathcal{H}''(a;b;(c,\ell);1) \\
&\leq \frac{1}{\kappa+2}.
\end{aligned}$$

Hence form Lemma 3, $\mathcal{H}(a;b;(c,\ell);z) \in \kappa - \mathcal{UCV}$. $\square$

For a function $f \in \mathcal{C}$ given by (1), the Alexander transform $A(f) : \mathbb{D}_1(0) \to \mathbb{C}$ is defined by (see Alexander (2015))

$$A(f)z = \int_0^z \frac{f(w)}{w} dw = z + \sum_{n=2}^{\infty} \frac{a_n}{n} z^n.$$

Theorem 10. *If $a, b, c, \ell \in \mathbb{R}$ with $0 < a < b$, $c \geq 1$ and $\ell \geq 1$. Then the sufficient condition for $A(\mathcal{H}(a; b; (c, \ell); z))$ to be in the class $\mathcal{S}^*$ is $\mathcal{H}(a; b; (c, \ell); 1) \leq 2$.*

Proof. From (4), we have

$$\frac{\mathcal{H}(a; b; (c, \ell); z)}{z} = 1 + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(b)_{n-1}(c)_{n-1}^{\ell(n-1)}} \frac{z^{n-1}}{(n-1)!} = 1 + \sum_{n=2}^{\infty} A_{n-1} z^{n-1},$$

where

$$A_{n-1} = \frac{(a)_{n-1}}{(b)_{n-1}(c)_{n-1}^{\ell(n-1)}(n-1)!}.$$

Thus,

$$\begin{aligned} A(\mathcal{H}(a; b; (c, \ell); z)) &= \int_0^z \frac{\mathcal{H}(a; b; (c, \ell); w)}{w} dw \\ &= z + \sum_{n=2}^{\infty} A_{n-1} \frac{z^n}{n} = \sum_{n=1}^{\infty} a_{n-1} z^n, \end{aligned}$$

where $a_1 = 1$, $a_n = \dfrac{A_{n-1}}{n}$, $n \geq 2$. From Lemma 2, we have $A(\mathcal{H}(a; b; (c, \ell); z)) \in \mathcal{S}^*(0) = \mathcal{S}^*$ if,

$$\sum_{n=2}^{\infty} n|a_n| \leq 1.$$

That is

$$\begin{aligned} \sum_{n=2}^{\infty} n|a_n| &= \sum_{n=2}^{\infty} n \frac{A_{n-1}}{n} \\ &= \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(b)_{n-1}(c)_{n-1}^{\ell(n-1)}(n-1)!} \\ &= \mathcal{H}(a; b; (c, \ell); 1) - 1 \leq 1. \end{aligned}$$

Which is true, since $\mathcal{H}(a; b; (c, \ell); 1) \leq 2$. This completes the proof. $\square$

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